An Integrated Approach to Measuring Asset and Liability Risks in Financial Institutions*

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Abstract

Risk measurement models for financial institutions typically focus on the net portfolio position and thus ignore distinctions between 1) assets and liabilities and 2) uncollateralized and collateralized liabilities. However, these distinctions are economically important. Liability risks affect the total amount of claims on the institution, while asset risks affect the amount available for claimants. Collateralization also affects the amounts recovered by different classes of claimants. We analyze a model of a financial institution with risky assets and liabilities, with potentially varying levels of collateralization across liabilities, showing that correct economic risk capital allocation requires complete segregation of asset, uncollateralized liability, and collateralized liability risks, with different risk measures for each. Our numerical analyses suggest that the conventional approach frequently yields over-investment in risky assets.

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1 Introduction

Risk models for financial institutions often focus on the net portfolio outcome and thus ignore the distinctions between asset and liability risks. For example, models treat short positions as "negative assets" with negative payoffs, and they treat swaps similarly by rolling negative payoff outcomes into the overall portfolio position. It is also common practice to ignore differences in the collateralization provisions in contracts with potential liabilities, as collateralization affects the allocation of assets among different claimants but does not affect the overall net portfolio position.

While these abstractions may help to make the problem of risk measurement more tractable, they present a misleading picture of the true economic costs of risk when viewed from the perspective of the firm's counterparties and regulators. To illustrate the importance of these distinctions, a decrease in the value of assets may be equivalent to an increase (of identical magnitude) in liabilities if one's only concern is the firm's book equity (i.e., the difference between assets and liabilities), but in an insolvency scenario the two changes are not equivalent in terms of their impact on claimants: A decrease in assets translates into less value being distributed among claimants, while an increase in liabilities reflects an increase in claims on the same set of assets. Moreover, the impact of insolvency on individual unsecured claimants will obviously depend crucially on the extent of collateralization in the firm's other contracts. Similarly, the final bill to a regulator or guarantor charged with covering the insured liabilities of a failed institution also depends on collateralization: if the institution's assets have already been pledged to (uninsured) counterparties, the guarantor will obviously have a much higher bill.

In practice, financial institutions have a mix of risky assets and risky liabilities and vary in their use of collateral support. Banks, for example, are often conceived on the blackboard as institutions with risky assets and fixed liability promises, though they create risky liabilities through derivative transactions; the main risks in insurance companies, on the other hand, are thought to be on the liability side of the balance sheet—though they also invest in risky assets. Collateralization of liabilities also varies. For example, the *ISDA Margin Survey 2015* reveals widespread use of collateral support in derivative contracts, although utilization is by no means universal within sectors and varies by type of financial institution. Thus, an economic approach to risk pricing, grounded either in the private costs of risk (by measuring the cost to the firm of imposing risk on its counterparties) or the social costs (by measuring the cost to the public of a bailout), must consider the distinctions between assets and liabilities as well as the use of collateralization.

In this paper, we develop a risk pricing model for a general financial institution with risky assets and risky liabilities in a single period setting. The institution is free to determine the riskiness of its asset portfolio and its exposures to liabilities; it is also free to determine whether the liabilities are collateralized or uncollateralized. It is constrained in its choices by the preferences of its risk-averse counterparties. We show that risk pricing and performance evaluation, such as that implemented through *Return on Risk-Adjusted Capital* (RAROC) calculations, requires completely separate treatment of assets and liabilities. Specifically, the risk allocation formulas for liabilities are different than those for assets, and the formulas for liabilities also vary according to the underlying degree collateralization. These results flow from the differing effects of asset, uncollateralized liability, and collateralized liability risks on counterparties.

We derive closed form pricing formulas for assets and for the two types of liabilities which have embedded risk allocations that can be used to derive risk measures and calculate RAROC values. When using risk measures, the correct measure for uncollateralized liabilities takes the log-exponential form derived in Bauer and Zanjani (2016), which also delivers the risk allocation formula derived by Ibragimov, Jaffee and Walden (2010) in the limiting case of complete markets. The measure for assets resembles tail value-at-risk (TVaR) or Expected Shortfall (ES) (Acerbi and Tasche, 2002). The correct measure for collateralized liabilities is a blending of the two. In both cases, the risk measure only penalizes outcomes in states of default, and outcomes are weighted by the relative values placed on recoveries in states of default. With both random assets and random liabilities, however, the default thresholds are also random, meaning that neither risk measure is necessarily monotonic.

We compare resulting marginal capital costs and optimal choices to the conventional approach, where a company is netting assets and liabilities and uses risk measures for risk capital allocation, in a detailed example setting. In particular, we consider allocations based on the most common risk measures—Value-at-Risk (VaR) and Expected Shortfall (ES)—for our comparisons.[1] We find that resulting allocations differ substantially, and that in many situations the conventional approach allocates more of the cost to liability risk, relative to the economic approach—and in turn less to asset risk. This implies that a company relying on risk measures for capital allocation will tend to over-invest in risky assets, and engage less in taking liability risk (e.g., selling insurance) than is economically efficient.

1.1 Relationship to the Literature

For financial practitioners, the asset pricing theory revolution of the 1960s and 1970s provided a critical framework for estimating the fair rate of return on capital. Implementation was not always straightforward, but the CAPM became the cost of capital workhorse for business generally (Brotherson et al., 2013; Bancel and Mitoo, 2014, e.g.) and particularly for pricing in regulated industries such as insurance (Hill, 1979, e.g.) and public utilities (Litzenberger, Ramaswamy, and Sosin, 1980, e.g.).

Some problems lingered, however. Financial institutions such as banks and insurance companies were engaged in multiple lines of business, dealing in illiquid and opaque products lacking well-defined market values. Attempts to assign values through bottom up fundamental analysis of the products—for example, by trying to estimate product betas—often yielded unstable and inconsistent results.[2] Moreover, only through blind luck would the (weighted) sum of the product line betas coincide with the overall beta of the institution. In other words, while top-down methods (e.g., based on the CAPM beta of a holding company stock) could be used to assess the risk of the organization as a whole, the practitioner was left with no theoretical guidance on how to allocate that risk to divisions and product lines.

RAROC and other risk-adjusted performance indicators rooted in capital allocation techniques emerged, in part, to address this line allocation problem. These measures, typically derived from economic capital models, continue to be ubiquitous in pricing and performance measurement applications in banking and insurance (McKinsey&Company, 2011; Society of Actuaries, 2016).

Given their pedigrees as *ad hoc* approaches, it is not surprising that capital allocation and RAROC have attracted skepticism. Inconsistencies emerge when using pricing approaches grounded

[1] ES is also refereed to as Conditional Tail Expectation (CTE), Tail-Value-at-Risk (TVaR), or Conditional Value-at-Risk (CVaR).

[2] See Cummins and Harrington (1985) and Cox and Rudd (1991) for examples of fruitless attempts to measure underwriting betas in insurance.

in standard asset pricing models using complete markets. Some question the necessity of capital allocation (Phillips, Cummins, and Allen, 1998; Sherris, 2006, e.g.). Others attempt reconciliation, identifying corrective adjustments for RAROC (Crouhy, Turnbull, and Wakeman, 1999; Perold, 2005; Milne and Onorato, 2012, e.g.) or alternatives (Erel, Myers, and Read, 2015). While these criticisms are valid, it is important to note that the standard asset pricing setup assumes away the complications that motivated capital allocation and RAROC in the first place. The latter techniques emerged because 1) valuation was not transparent and 2) frictional costs were important. Indeed, if markets were frictionless and complete, there would be no need for capital allocation nor for intermediaries. Allocation of capital is necessary only when there is an associated frictional cost that requires allocation. Moreover, while a frictional cost of capital may motivate capital allocation, it does not resolve the paradox of the intermediaries existence. To elaborate, if customers and counterparties have access to complete markets, then it becomes inefficient to use an intermediary burdened with frictional costs (see, e.g., Ibragimov, Jaffee and Walden (2010)).

Recognition of these complications led Froot and Stein (1998) to study intermediary pricing in an incomplete markets setting with frictional financing costs. This approach produced an intermediary-specific pricing function with an unclear connection to capital allocation and RAROC. Froot and Stein criticized the use of arbitrary risk measures in risk pricing, emphasizing instead the connection between pricing and the real-world objectives and constraints faced by the institution. Other papers following this line recover explore pricing with various frictional assumptions, uncovering connections to RAROC and capital allocation (Stoughton and Zechner, 2007; Bauer and Zanjani, 2016, e.g.).

A parallel literature focuses less on the financial economic foundation of capital allocation and more on the technical aspects of allocation. Embedded in this approach is the notion of portfolio optimization constrained by a risk measure (Tasche, 2004; Bauer and Zanjani, 2013, e.g.). The two primary concerns of the literature are 1) the risk capital allocation implied by the risk measure (Denault, 2001; Kalkbrener, 2005; Powers, 2007, e.g.) and the proper form of the risk measure (Artzner et al., 1999; Acerbi, 2002; Acerbi and Tasche, 2002, e.g.).

By reducing risks to random variables, risk measure mathematics eliminate distinctions between assets and liabilities. Theoretical papers are often silent on the topic. Theory typically focuses on the net portfolio position, with the component random variables representing the profit or loss on a sub-portfolio. Applications often adopt the conventions of the institution of interest. Historically, insurance companies have been primarily concerned with the pricing of liabilities, while banks have been concerned with the pricing of assets. Thus, papers on insurance typically focus on allocating capital to liabilities (Myers and Read, 2001; Tsanakas and Barnett, 2003, e.g.) while bank applications are focused on assets (Kalkbrener, 2005; Erel, Myers, and Read, 2015, e.g.). In reality, however, financial institutions have both risky assets and risky liabilities, and it seems natural to account for both types of risk. Yet the approach to doing so can lead to a practical conundrum that is alluded to by D'Arcy (2011, pp. 144-145):

> "...robust capital allocation method focuses on aggregate portfolio risk, including both underwriting and investment operations. Any decline in the value of an investment generates the same type of call on capital that an underwriting loss would for a line of business, and should be treated similarly in the capital allocation process. Other capital allocation methods tend to focus on allocating capital to lines of business. [...] In these methods investment risk is either considered to be a component of a line of busi-

> ness or eliminated by assuming risk-free investment returns. While this approach may be acceptable for pricing, it is not appropriate for risk management or performance evaluation applications of capital allocation. An insurer needs to maintain capital to support risky investments in the same way that capital needs to be allocated to a line of business that generates risk. If an investment portfolio takes on additional risk, then additional capital needs to be allocated to support this new position."

D'Arcy's conflicted endorsement of assuming away asset risk for pricing purposes connects to the original motivation of capital allocation in insurance. Insurers typically view themselves as price-takers on asset purchases but price-makers on insurance contract sales. The primary need for capital allocation, as noted above, was to allocate residual capital costs to insurance contract prices. If capital is allocated to assets—this goal is seemingly jeopardized. An integrated approach is needed.

Our theoretical approach, like many others, is rooted in the fundamental assumption that risk management is driven by credit quality concerns. However, an important distinction is that we do not model these concerns in reduced form: We instead model the micro-foundations of the credit quality concerns by including the utilities of individual claimants and counterparties in the form of participation constraints.[3] This choice has an important consequence. The fundamental concern of each claimant is the nature of the repayment from the institution; in states of default, the critical concern is the amount of recovery. This fact drives the central pricing results of the mode—various types of risk are penalized or rewarded based on their impact on claimant recoveries in states of default—and also drives the specific forms of risk measure that we obtain for asset and liability risks.[4]

1.2 A Simple Example

Consider two institutions contemplating positions in two securities. The first is a risk-free asset with an interest rate of zero percent. The second is a risky "zero beta" stock currently trading at \$100 that can either double to \$200 or fall to \$0 with equal proability.

Institution A starts with \$200 in debt and \$100 in equity. It invests \$100 in the risk-free asset and buys two shares of the stock for \$200.

Institution B also starts with \$200 in debt and \$100 in equity. It decides to take a \$200 short position in the stock by borrowing two shares and selling them. It invests its cash, a total of \$500 (composed of \$200 in deposits, \$100 in equity, and \$200 in short sale proceeds), in the risk-free asset.

It is easy to verify that the positions of the two institutions have similar risk when focusing on the net payoff. If the stock goes up, Institution A's asset portfolio is worth $\$100+\$200\times 2 = \$500$, leaving equity of \$300 after depositors are paid. If the stock goes down, Institution A's asset portfolio is worth $\$100 + \$0 \times 2 = \$100$; it enters bankruptcy and has equity of $-\$100$. Institution B, on the other hand, is left with \$500 of assets and \$300 of equity if the stock goes down, and a portfolio worth \$100 (net of the obligation to securities lenders) with equity of $-\$100 = \$500 -$

[3]This approach is thus similar in form to that of Zanjani (2002) and Bauer and Zanjani (2016).

[4]The choice also allows for flexibility in determining values in the sense that we can apply a complete markets approach (in which case the solution for liability allocations collapses to that offered by Ibragimov, Jaffee and Walden (2010)) or an incomplete markets approach.

$\$200 - \$200 \times 2$ if the stock goes up. Indeed, if we ignore the detail that the short position represents a liability, and instead follow the widespread practice of treating it as a "negative asset," the portfolios of Institutions A and B are mirror images of each other and identical from a risk perspective.

When contemplating the bankruptcies, however, the validity of the analogy comes into question and depends on the details. Institution A's debtholders receive 50 cents $= \$100/\$200$ on the dollar in the event of bankruptcy. What Institution B's debtholders receive depend on the securities lending arrangement. If the securities lenders are unsecured creditors and *pari passu* with the debtholders, then debtholders receive 83 cents $= \$500/\$600$ on the dollar in the event of bankruptcy. If the securities lenders are effectively secured by collateral, however, then they are shielded from bankruptcy and receive full payment, while the original debtholders are back to receiving 50 cents $= \$100/\$200$ on the dollar.

This example illustrates two simple but important points when contemplating an economic approach to risk pricing. First, liability risk is not the same as asset risk in terms of its economic effects on counterparties and requires fundamentally different treatment. Second, collateralization provisions also determine the economic impact of liability risks, with the impact varying across secured and unsecured creditors; the latter in particular can experience significant negative externalities from provisions that secure the former. In what follows, we develop these points in the context of a general model of risk pricing that considers both asset-side and liability-side risks.

2 Capital Allocation with Risky Assets and Risky Liabilities: A General Model

2.1 Model with Uncollateralized Liabilities

Formally, we consider a company that has N potential claimants, with the company's future (and possibly random) obligation to claimant i denoted by:

$$I_i = I_i(L_i, q_i) = q_i \times L_i.$$

where L_i is modeled as a continuous, non-negative random variable with (joint) density function $f_{L_1,L_2,\ldots,L_N}$ and q_i represents the company's chosen level of exposure to claimant i. Note that this form is flexible enough to represent straight debt contracts (by setting a degenerate marginal distribution for L_i, so that $L_i = 1$), quota share insurance contracts (where $q_i \in [0,1]$) as well as the potential liability in a swap contract. One could also contemplate generalized forms of $I_i(L_i, q_i)$, possibly with multiple dimensions to the exposure variable, that would allow for nonlinear obligation structures (such as insurance deductibles and policy limits), but we focus on the linear case for tractability. The claims are sold at prices p_i,which represent amounts collected now—such as borrowings or insurance premiums—in exchange for the promises of future payments I_i.

On the asset side, the company takes the sum of revenues from the sale of financial claims, $\sum p_i$, as well as any additional equity, and has the opportunity to invest in M risky projects with random returns S_j at amounts a_j. The company's portfolio is thus a random variable defined as:

$$A = a(1 + \bar{S}) = \sum_{i=1}^{M} a_j(1 + S_j)$$

If total claims are less than company assets, all are paid in full. If not, all claimants are paid at the same rate per dollar of coverage. The total claims submitted are:

$$I = I(L_1, L_2, \ldots, L_N, q_1, q_2, \ldots, q_N) = \sum_{j=1}^{N} I_j(L_j, q_j),$$

and we define the claimant's recovery as:

$$R_i = \min\left\{ I_i(L_i, q_i), \frac{A}{I}\, I_i(L_i, q_i) \right\}. \tag{1}$$

Accordingly, $\{I > A\}$ denote the states in which the company defaults whereas $\{I \leq A\}$ are the solvent states.

We use von Neumann-Morgenstern expected utility to represent risk averse claimant preferences, as in:

$$v_i(a_1, ..., a_M, W_i - p_i, q_1, ..., q_N) = \mathbb{E}\left[U_i\left(W_i - p_i + R_i - L_i\right)\right],$$

where W_i is the (possibly random) wealth of the claimant.

Given a utility function V company then chooses investments, liability exposures, and prices for the liabilities to maximize

$$\max_{\{a_j\},\{q_i\},\{p_i\}} \mathbb{E}V\left(\sum p_i - \sum R_i + \sum_{j=1}^{M} a_j(1+S_j) - \sum_{j=1}^{M} a_j - \tau\left[\sum_{j=1}^{M} a_j - \sum p_i\right]\right), \tag{2}$$

subject to participation constraints for potential claimants:

$$v_i(a_1, ..., a_M, W_i - p_i, q_1, ..., q_N) \geq \gamma_i \quad \forall i. \tag{3}$$

and to nonnegativity constraints on the asset and liability exposure choices.

Two comments on the setup are warranted. First, the objective function is flexible with respect to specifying company preferences. It of course fits naturally with expected utility maximization. It can also accommodate "market-consistent" valuation by setting:

$$\mathbb{E}V = \frac{1}{1+r_f}\mathbb{E}^{\mathbb{Q}}\left(\sum p_i - \sum R_i + \sum_{j=1}^{M} a_j(1+S_j) - \sum_{j=1}^{M} a_j - \tau\left[\sum_{j=1}^{M} a_j - \sum p_i\right]\right)$$

i.e., the discounted expectation calculated under the risk neutral measure $\mathbb{Q}$. It should be noted, however, the concept of market-consistent valuation is slippery here as we allow for non-traded assets (for example, insurance contracts) that lack well-defined market prices. Moreover, we will generally assume that markets are incomplete: An assumption of complete markets would lead to obvious paradoxes within the model, particularly by removing the economic need for an intermediary and by trivializing risk management.

Second, similar discussion could be applied to the claimant utility functions. It should be noted that the model also incorporates the possibility of claimants accessing the market for traded assets via the random wealth component, which allows them the possibility of (partially) hedging default exposures to the company. Optimal hedging can be explicitly incorporated by modeling claimant investments as choice variables (see Bauer and Zanjani (2016)), which does not affect the form of the solution that we study below.

The first order conditions for this problem (when the nonnegativity constraints for assets and liability exposures do not bind) can be expressed as:

$$\begin{aligned}
[a_j] \qquad & \mathbb{E}\left[V'\left(S_j - \sum_k \frac{\partial R_k}{\partial a_j} - \tau\right)\right] + \sum_k \lambda_k \frac{\partial v_k}{\partial a_j} = 0, \\
[q_i] \qquad & \lambda_i \frac{\partial v_i}{\partial q_i} - \mathbb{E}\left[V'\left(\sum_k \frac{\partial R_k}{\partial q_i}\right)\right] + \sum_{k \neq i} \lambda_k \frac{\partial v_k}{\partial q_i} = 0, \\
[p_i] \qquad & \mathbb{E}\left[V'\left((1+\tau)\right)\right] - \lambda_i\, v_i' = 0,
\end{aligned}$$

where λ_k designate the Lagrange multipliers on the participation constraints.

The first order condition for q_i embeds costs and benefits that are borne directly by the company and claimant i, as well as effects on other claimants that create indirect costs for the company. These latter effects, captured in the third term, derive from the fact that all claimants are rivals for the same assets in bankruptcy, and an increase in *ex ante* exposure to one claimant decreases the recoveries of other claimants in insolvency scenarios whenever a *pro rata* bankruptcy rule such as (1) is employed.

Similarly, the optimality condition for a_j trades off a direct marginal cost for the firm (the first term on the left-hand side—composed of the sum of marginal increase in payments to claimants and the frictional cost of capital, net the return on the asset) with an indirect benefit related to effect of the asset on counterparties. The latter benefit relates to the relaxation of the participation constraints, as claimants enjoy increased recoveries in bankruptcy scenarios with the increase in the asset.

These "indirect" effects on claimants are the fundamental economic factors that cause risk pricing and hurdle rates to deviate from simple calculations of expected gain and loss. We study them further below by working with the optimality conditions.

Working with the optimality conditions, we obtain the following key marginal risk pricing expression for liabilities at the optimum:

$$(1+\tau)\frac{\partial p_i^*}{\partial q_i} = \mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}}\frac{\partial I_i}{\partial q_i}\right] + \underbrace{\frac{\mathbb{E}\left[\mathbf{1}_{\{I>a\}}\ \sum_k \frac{U_k'}{v_k'}\ \frac{I_k}{I}(1+\bar{S})\frac{\partial I_i/\partial q_i}{I}\right]}{\mathbb{E}\left[\mathbf{1}_{\{I>a\}}\ \sum_k \frac{U_k'}{v_k'}\ \frac{I_k}{I}(1+\bar{S})\right]}}_{=\phi_i^q}\ \left[\tau - \mathbb{E}^V[\mathbf{1}_{\{I\leq A\}}\bar{S} - \mathbf{1}_{\{I>A\}}]\right]a. \tag{4}$$

where we are writing $\mathbb{E}^V[X] \equiv \mathbb{E}[\frac{V'}{\mathbb{E}V'}X]$ (i.e., the transformed measure has density $\frac{V'}{\mathbb{E}V'}$ with respect to the physical probability measure).

The price per unit exposure is on the left hand side is set equal to the marginal cost of risk. The marginal price response on the left hand side is adjusted by the factor $(1+\tau)$ to reflect the fact that liability premiums substitute for costly capital; if the frictional cost τ were applied to all assets rather than just to capital, this adjustment term would disappear. The marginal cost of risk exposure on the right-hand side is decomposed into an modified marginal actuarial cost (calculated under a modified measure reflecting the company's risk aversion) plus a risk penalty reflecting the impact of the exposure on the recoveries of counterparties. Since $\sum \phi_i^q q_i = 1$, this risk penalty can be interpreted as an allocation of total assets to claimant i times a marginal cost of assets (which is the frictional cost of capital minus the company's valuation of portfolio return).

We also obtain the following risk pricing expression for assets:

$$\mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}}S_j - \mathbf{1}_{\{I>A\}}\right] = \tau - \underbrace{\frac{\mathbb{E}\left[\mathbf{1}_{\{I>a\}} \sum_k \frac{U_k'}{v_k'}\frac{I_k}{I}(1+S_j)\right]}{\mathbb{E}\left[\mathbf{1}_{\{I>a\}} \sum_k \frac{U_k'}{v_k'}\frac{I_k}{I}(1+\bar{S})a\right]}}_{=\phi_i^a}\left[\tau - \mathbb{E}^V[\mathbf{1}_{\{I\leq A\}}\bar{S} - \mathbf{1}_{\{I>A\}}]\right]a. \tag{5}$$

The company's valuation of the expected return is on the left hand side. The marginal cost of investing in the asset decomposes into a base frictional cost plus a risk subsidy reflecting the impact of the asset exposure on counterparty recoveries. Assets generate returns in states of default, which are valued by counterparties, and so actually receive a risk *subsidy* in this formulation. Like the risk penalty in (4), since $\sum \phi_i^a a_j = 1$ the risk subsidy for the total investment in asset j can be interpreted as an asset allocation times the same marginal asset cost as in (4).

Alternatively, one can also re-express the asset condition in terms of the asset's return in excess of an investment in a risk-free asset:

$$\mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}}\left[S_j - r_f\right]\right] = \bar{\phi}_j\,\mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}}\left[\bar{S} - r_f\right]\right]a, \tag{6}$$

where

$$\bar{\phi}_j = \frac{\sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{I>A\}}\,U_k'\,\frac{I_k}{I}\left[r_f - S_j\right]\right]}{v_k'}}{\sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{I>A\}}\,U_k'\,\frac{I_k}{I}\left[ar_f - \sum a_j S_j\right]\right]}{v_k'}}.$$

in which case a security could have a risk penalty or a risk subsidy, with the size and sign of the penalty/subsidy being determined by the performance of the security relative to the risk-free security in states of default. Since $\sum \bar{\phi}_j a_j = 1$, this penalty also can be interpreted as a share of the total cost of assets, with the cost in this case being the expected excess return on the company's portfolio over a risk-free investment—so that the cost of investment amounts to the foregone excess return opportunity when diverting resources toward asset j.

These theoretical results, derived from a model where the economic motivation for risk management flows from concerns with external financing provided by creditors and counterparties, involve some significant deviations from practical approaches to risk pricing. The economic approach implies that both asset and liability risks must be considered in a risk pricing model and, moreover, that they require completely separate treatment in a capital allocation model. This differs from common practice in important ways.

In general, asset and liability risks are typically not segregated in risk pricing and performance measurement models. A typical approach in banking is based on the application of a risk measure to a combined asset and liability portfolio.[5] In insurance, asset risks are often ignored, and the goal of the risk pricing exercise is to allocate capital exclusively to liability risks. As shown above, both

[5] See Tasche (2000) for an early (and influential) example of this logic.

asset and liability exposures have risk effects on counterparties that require consideration. Some instruments, such as swaps or forwards, may even have asset *and* liability characteristics, requiring division of instrument into asset and liability components during risk analysis. Furthermore, it is evident when comparing (4) and (5) that, in the aggregate, the risk penalties for liabilities are exactly offset by the risk subsidies for assets. Thus, assets and liabilities are of equal importance in the risk pricing; neglecting either asset risk subsidies or liability risk penalties could imply significant evaluation errors.

Also important is the cost being allocated when calculating the risk penalty. Common RAROC and RORAC implementations in banking and insurance compare a target ROE an exposure's return divided by a capital allocation, which is typically a target ROE. However, even if we interpret τ as a target ROE, the cost being allocated nets out the return on the asset portfolio from the target return on capital. For the liability exposures, pricing based on (4) indicates that, in aggregate, indirect costs amounting to

$$\tau\left(a-\sum p_i\right)-\mathbb{E}^V[\mathbf{1}_{\{I\le A\}}\bar{S}-\mathbf{1}_{\{I>A\}}]a$$

are recovered.[6] The first term is simply the total required return on capital, as expected, but the second term is the total expected return on the asset portfolio, which is subtracted. On the asset side, note that the relevant hurdle rate for an investment from (6) is keyed to the average portfolio excess return, not the target return on equity.

Finally, since assets and liabilities affect counterparty recoveries in different ways, the appropriate risk measurement tools must be tailored for each. Much literature has been devoted to analyzing desirable mathematical properties of risk measures (e.g., coherence or convexity), and these properties are often relied upon by practitioners in selecting risk measures for pricing and performance measurement. However, the risk measures implied by the economic model for assets and liabilities are not the same, and neither adhere to the mathematical properties typically imposed on risk measures.

As derived in Bauer and Zanjani (2016), the risk measure appropriate for liabilities—in the sense of having a gradient that delivers the correct marginal cost of risk as implied by (4):

$$\rho(I)=\exp\left(\mathbb{E}^{\widetilde{\mathbb{P}}}\left[\log\left(I\right)\right]\right)$$

where the transformed measure is related to the physical probability measure $\mathbb{P}$ by the density:

$$\frac{\partial\widetilde{\mathbb{P}}}{\partial\mathbb{P}}=\frac{\mathbf{1}_{\{I>A\}}\ \sum_k\frac{U_k'}{v_k'}\frac{I_k}{I}(1+\bar{S})}{\mathbb{E}\left[\mathbf{1}_{\{I>A\}}\sum_k\frac{U_k'}{v_k'}\frac{I_k}{I}(1+\bar{S})\right]},$$

so that the probability of a state is weighted in relation to the values placed by counterparties on an additional unit of invested assets in that state.

The risk measure whose gradient yields the appropriate marginal cost of risk for assets, on the other hand, can shown to be a conditional expectation calculated under a transformed measure:

$$\rho\left(A\right)=\mathbb{E}^{\widetilde{\widetilde{\mathbb{P}}}}\left[A|I>A\right]$$

[6]This assumes either that the pricing functions are constructed to be linear or that they are integrable, with the allocation ratios per unit of exposure held constant.

=where the transformed measure is related to the physical probability measure $\mathbb{P}$ by the density:

$$\frac{\partial\widetilde{\mathbb{P}}}{\partial\mathbb{P}} = \frac{\mathbf{1}_{\{I>A\}}\ \sum_k \frac{U'_k}{v'_k}\frac{I_k}{I}}{\mathbb{E}\left[\mathbf{1}_{\{I>A\}}\ \sum_k \frac{U'_k}{v'_k}\frac{I_k}{I}\right]},$$

so that probabilities are weighted in relation to the values placed by counterparties on an additional dollar of recovery in each state. Only in the case where liabilities are nonrandom will this measure obey the coherence axioms, in which case the measure will amount to a spectral version of Expected Shortfall.[7]

2.2 Model with Uncollateralized and Collateralized Liabilities

Formally, we consider a company that has N^u potential uncollateralized claimants, with the company's future (and possibly random) obligation to claimant i denoted by

$$I_i^u = I_i^u(L_i^u, q_i^u) = q_i^u \times L_i^u.$$

where L_i^u is modeled as a continuous, non-negative random variable and q_i^u represents the company's chosen level of exposure to claimant i. Note that this form is again flexible enough to represent straight debt contracts (by setting a degenerate marginal distribution for L_i^u, so that $L_i^u = 1$), quota share insurance contracts (where $q_i^u \in [0,1]$) as well as the potential liability in a swap contract. One could also contemplate generalized forms of $I_i^u(L_i^u, q_i^u)$, possibly with multiple dimensions to the exposure variable, that would allow for nonlinear obligation structures (such as insurance deductibles and policy limits), but we focus on the linear case for tractability. Collateralized claimants are represented similarly, except with different superscripts:

$$I_i^c = I_i^c(L_i^c, q_i^c) = q_i^c \times L_i^c.$$

Liabilities are assumed to be distributed according to the (joint) density function:

$$f_{L_1^u, L_2^u, \ldots, L_N^u, L_1^c, L_2^c, \ldots, L_N^c}.$$

Uncollateralized and collateralized claims are sold at prices p_i^u and p_i^c, respectively, which represent amounts collected now—such as borrowings or insurance premiums—in exchange for the promises of future payments I_i^u and I_i^c.

On the asset side, the company takes the sum of revenues from the sale of financial claims, $\sum p_i^u$ and $\sum p_i^c$, as well as any additional equity, and has the opportunity to invest in M risky projects with random returns S_i at amounts a_i. The company's portfolio is thus a random variable and is again defined as:

$$A = a(1+\bar{S}) = \sum_{j=1}^{M} a_i(1+S_i).$$

[7]See Acerbi (2002) for the analysis of spectral risk measures. Bauer and Zanjani (2015) analyze a simplified model with no liability risk, finding spectral ES to be the appropriate risk measure.

If total claims are less than company assets, all are paid in full. If not, all claimants are paid at the same rate per dollar of coverage. The total claims submitted are:

$$I = \sum_{i=1}^{N^u} I_i^u(L_i^u, q_i^u) + \sum_{i=1}^{N^c} I_i^c(L_i^c, q_i^c) = I^u + I^c.$$

The amount actually recovered depends on whether the claim is collateralized or uncollateralized. For collateralized claims, we simplify by assuming that collateralized claims are always senior to uncollateralized claims and *pari passu* with other collateralized claims. Thus, in bankruptcy, the collateralized claims are paid in full if assets are sufficient to meet the collateralized obligations. If assets are not sufficient to cover collateralized claims, assets are divided up among collateralized claimants on a *pro rata* basis, and uncollateralized claimants receive nothing. We can then express the collateralized recovery for the i-th collateralized claimant as:

$$R_i^c = \min\left\{ I_i^c(L_i^c, q_i^c), \frac{A}{I^c}\, I_i^c(L_i^c, q_i^c) \right\}. \tag{7}$$

For an uncollateralized claim, the recovery in a bankruptcy situation is similar, but based on what (if anything) remains after collateralized claims have been paid:

$$R_i^u = \min\left\{ I_i^u(L_i^u, q_i^u), \frac{A - I^c}{I^u}\, I_i^u(L_i^u, q_i^u) \right\}. \tag{8}$$

Accordingly, $Z = \{I \leq A\}$ are the solvent states, and we have two types of bankruptcy situations. In the first, $D^u = \{I^c < A < I\}$, collateralized claims are paid in full, and what remains is shared among uncollateralized claimants. In the second, $D^c = \{A < I^c\}$, assets are divided among collateralized claimants, while the uncollateralized claimants receive nothing.

We use VNM expected utility to represent risk averse claimant preferences, as in:

$$v_i^u(a_1, ..., a_M, W_i - p_i^u, q_1^u, ..., q_N^u, q_1^c, ..., q_N^c) = \mathbb{E}\left[U_i^u\left(W_i - p_i^u + R_i^u, L_i^u\right)\right]$$

and

$$v_i^c(a_1, ..., a_M, W_i - p_i^c, q_1^c, ..., q_N^c) = \mathbb{E}\left[U_i^c\left(W_i - p_i^c + R_i^c, L_i^c\right)\right]$$

The company then chooses investments, liability exposures, and prices to maximize

$$\max_{\{a_j\},\{q_i^c\},\{q_i^u\},\{p_i^c\},\{p_i^u\}} V\left(\begin{array}{l} \sum p_i^u + \sum p_i^c - \sum R_i^u - \sum R_i^c \\ \qquad + \sum_{j=1}^M a_j(1+S_j) - \sum_{j=1}^M a_j - \tau\left[\sum_{j=1}^M a_j - \sum p_i\right] \end{array}\right), \tag{9}$$

subject to participation constraints for claimants:

$$\begin{array}{ll} [\lambda_i^u] & v_i^u(a_1, ..., a_M, W_i - p_i^u, q_1^u, ..., q_N^u, q_1^c, ..., q_N^c) = \mathbb{E}\left[U_i^u\left(W_i - p_i^u + R_i^u, L_i^u\right)\right] \geq \gamma_i^u \ , \\ [\lambda_i^c] & v_i^c(a_1, ..., a_M, W_i - p_i^c, q_1^c, ..., q_N^c) = \mathbb{E}\left[U_i^c\left(W_i - p_i^c + R_i^c, L_i^c\right)\right] \geq \gamma_i^c \ , \end{array}$$

and to a budget constraint and non-negativity constraints for investment and liability exposure choices.

The first order conditions for this problem can be expressed as:

$$[a_j] \qquad \mathbb{E}\left[V'\left(S_j - \sum_k \frac{\partial R_k^u}{\partial a_j} - \sum_k \frac{\partial R_k^c}{\partial a_j} - \tau\right)\right] + \sum_k \lambda_k^u \frac{\partial v_k^u}{\partial a_j} + \sum_k \lambda_k^c \frac{\partial v_k^c}{\partial a_j} = 0,$$

$$[q_i^u] \qquad \lambda_i^u \frac{\partial v_i^u}{\partial q_i^u} - \mathbb{E}\left[V'\left(\sum_k \frac{\partial R_k^u}{\partial q_i^u}\right)\right] + \sum_{k\neq i} \lambda_k^u \frac{\partial v_k^u}{\partial q_i^u} = 0, \tag{10}$$

$$[q_i^c] \qquad \lambda_i^c \frac{\partial v_i^c}{\partial q_i^c} - \mathbb{E}\left[V'\left(\sum_k \frac{\partial R_k^u}{\partial q_i^c} + \sum_k \frac{\partial R_k^c}{\partial q_i^c}\right)\right] + \sum_k \lambda_k^u \frac{\partial v_k^u}{\partial q_i^c} + \sum_{k\neq i} \lambda_k^c \frac{\partial v_k^c}{\partial q_i^c} = 0, \tag{11}$$

$$[p_i^u] \qquad \mathbb{E}\left[V'\left((1+\tau)\right)\right] - \lambda_i^u \, v_i^{u\prime} = 0,$$

$$[p_i^c] \qquad \mathbb{E}\left[V'\left((1+\tau)\right)\right] - \lambda_i^c \, v_i^{c\prime} = 0. \tag{12}$$

Rewrite $[a_j]$ as:

$$[a_j] \qquad \mathbb{E}^V[S_j \mathbf{1}_{\{I\leq A\}} - \mathbf{1}_{\{I>A\}}] - \tau + \sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} \, U_k^{u\prime} \, \frac{I_k^u}{I^u} \, [1+S_j]\right]}{v_k^{u\prime}} + \sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} \, U_k^{c\prime} \, \frac{I_k^c}{I^c} \, [1+S_j]\right]}{v_k^{c\prime}} = 0$$

Working with these equations, we obtain the following key risk pricing expression for liabilities, as the price per unit exposure is decomposed into an actuarial value plus a capital allocation times capital cost (which is the frictional cost of assets net of the portfolio return):

$$\mathbb{E} v_i^{u\prime} \frac{\partial p_i^{u*}}{\partial q_i^u} = \frac{\partial v_i^u}{\partial q_i^u}, \; \mathbb{E} v_i^{c\prime} \frac{\partial p_i^{c*}}{\partial q_i^c} = \frac{\partial v_i^c}{\partial q_i^c}, m \tag{13}$$

and:

$$(1+\mu)\frac{\partial p_i^{u*}}{\partial q_i^u} = \mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}} \, \frac{\partial I_i^u}{\partial q_i^u}\right] + \underbrace{\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} \, \frac{\partial I_i^u / \partial q_i^u}{I^u} \, \sum_k \frac{U_k^{u\prime}}{v_k^{u\prime}} \, \frac{I_k^u}{I^u}[a(1+\bar{S}) - I^c]\right]}{\sum \mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} \, U_k^{u\prime} \, \frac{I_k^u}{I^u} \, a(1+\bar{S})\right] + \sum \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} \, U_k^{c\prime} \, \frac{I_k^c}{I^c} \, a(1+\bar{S})\right]}}_{=\tilde{\phi}_i^{q^u}} \times \left[\tau - \mathbb{E}^V[\mathbf{1}_{\{I\leq A\}}\bar{S} - \mathbf{1}_{\{I>A\}}]\right]a, \tag{14}$$

$$(1+\mu)\frac{\partial p_i^{c*}}{\partial q_i^c} = \mathbb{E}^V\left[\mathbf{1}_{\{I\leq A\}} \, \frac{\partial I_i^c}{\partial q_i^c}\right] + \underbrace{\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} \, \sum_k \frac{U_k^{u\prime}}{v_k^{u\prime}} \, \frac{I_k^u}{I^u}[\partial I_i^c / \partial q_i^c]\right] + \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} \, \frac{\partial I_i^c / \partial q_i^c}{I^c} \, \sum_k \frac{U_k^{c\prime}}{v_k^{c\prime}} \, \frac{I_k^c}{I^c}[a(1+\bar{S})]\right]}{\sum \mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} \, U_k^{u\prime} \, \frac{I_k^u}{I^u} \, a(1+\bar{S})\right] + \sum \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} \, U_k^{c\prime} \, \frac{I_k^c}{I^c} \, a(1+\bar{S})\right]}}_{=\tilde{\phi}_i^{q^c}} \times \left[\tau - \mathbb{E}^V[\mathbf{1}_{\{I\leq A\}}\bar{S} - \mathbf{1}_{\{I>A\}}]\right]a. \tag{15}$$

It is easily verified that adding up is preserved:

$$\sum \phi_i^{q^u} q_i^u + \sum \phi_i^{q^c} q_i^c = 1,$$

so that we may interpret the risk penalties as allocations of capital costs. However, when comparing uncollateralized exposures with collateralized exposures, it is evident that the latter have a greater marginal impact on uncollateralized claimants since the uncollateralized exposure effect is scaled by $\frac{[a(1+\bar{S})-I^c]}{I^u} < 1$.

The differing effects of uncollateralized versus collateralized liabilities requires segregation of the two liability types in risk measurement. The risk measure that delivers the correct marginal costs for liability exposures takes the form of:

$$\rho(I) = \exp\left(\mathbb{E}\left[W^u \log\left(I^u\right)\right]\right) + \exp\left(\mathbb{E}\left[W^{c1} \log\left(I^c\right)\right]\right) + \mathbb{E}[W^{c2} I^c]$$

where each component features a weighting function that is fixed based on the optimal choices solving (9).

$$\begin{aligned}
W^u &= \frac{\mathbf{1}_{\{I^c<A<I\}} \sum_k \frac{U_k^{u\prime}}{v_k^{u\prime}} \frac{I_k^u}{I^u}[a(1+\bar{S}) - I^c]}{\sum \mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} U_k^{u\prime} \frac{I_k^u}{I^u} a(1+\bar{S})\right] + \sum \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} U_k^{c\prime} \frac{I_k^c}{I^c} a(1+\bar{S})\right]}, \\
W^{c1} &= \frac{\mathbf{1}_{\{A<I^c\}} \sum_k \frac{U_k^{c\prime}}{v_k^{c\prime}} \frac{I_k^c}{I^c}[a(1+\bar{S})]}{\sum \mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} U_k^{u\prime} \frac{I_k^u}{I^u} a(1+\bar{S})\right] + \sum \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} U_k^{c\prime} \frac{I_k^c}{I^c} a(1+\bar{S})\right]}, \\
W^{c2} &= \frac{\mathbf{1}_{\{I^c<A<I\}} \sum_k \frac{U_k^{u\prime}}{v_k^{u\prime}} \frac{I_k^u}{I^u}}{\sum \mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} U_k^{u\prime} \frac{I_k^u}{I^u} a(1+\bar{S})\right] + \sum \mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} U_k^{c\prime} \frac{I_k^c}{I^c} a(1+\bar{S})\right]}.
\end{aligned}$$

The first component of the risk measure evaluates the impact of uncollateralized liabilities on uncollateralized claimants (uncollateralized liabilities do not affect collateralized claimants since the former are assumed to be paid after tha latter). The second component evaluates the impact of collateralized liabilities on other collateralized claimants and takes a form similar to the first component. The final component evaluates the impact of collateralized exposures on uncollateralized claimants. The correct risk measure for marginal pricing thus takes the form of a weighted hybrid of the liability risk measure derived in Bauer and Zanjani (2016) (for the first two components) and a conditional expected value (for the last component).

We also obtain the following risk pricing expression for assets, again decomposing into a base cost (simply the frictional cost of capital) plus a capital allocation times capital cost (which is the frictional cost of capital minus the expected portfolio return):

$$\begin{aligned}
\mathbb{E}^V[S_j \mathbf{1}_{\{I\leq A\}} - \mathbf{1}_{\{I>A\}}] &= \tau - \underbrace{\frac{\sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} U_k^{u\prime} \frac{I_k^u}{I^u}[1+S_j]\right]}{v_k^{u\prime}} + \sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} U_k^{c\prime} \frac{I_k^c}{I^c}[1+S_j]\right]}{v_k^{c\prime}}}{\sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}} U_k^{u\prime} \frac{I_k^u}{I^u}\left[1+\bar{S}\right]a\right]}{v_k^{u\prime}} + \sum_k \frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}} U_k^{c\prime} \frac{I_k^c}{I^c}\left[1+\bar{S}\right]a\right]}{v_k^{c\prime}}}}_{=\phi_i^a} \\
&\qquad \times\left[\tau - \mathbb{E}^V[\mathbf{1}_{\{I\leq A\}}\bar{S} - \mathbf{1}_{\{I>A\}}]\right]a, \qquad (16)
\end{aligned}$$

Again, we can re-express the asset condition in terms of the asset's return in excess of an investment in a risk-free asset:

$$\mathbb{E}^V\left[\mathbf{1}_{\{I\le A\}}\left[S_j-r_f\right]\right]=\bar{\phi}_j\,\mathbb{E}^V\left[\mathbf{1}_{\{I\le A\}}\left[\bar{S}-r_f\right]\right]a, \tag{17}$$

where

$$\bar{\phi}_j=\frac{\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}}\,U_k^{u\prime}\,\frac{I_k^u}{I^u}\left[r_f-S_j\right]\right]}{v_k^{u\prime}}+\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}}\,U_k^{c\prime}\,\frac{I_k^c}{I^c}\left[r_f-S_j\right]\right]}{v_k^{c\prime}}}{\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}}\,U_k^{u\prime}\,\frac{I_k^u}{I^u}\left[ar_f-\sum a_jS_j\right]\right]}{v_k^{u\prime}}+\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}}\,U_k^{c\prime}\,\frac{I_k^c}{I^c}\left[ar_f-\sum a_jS_j\right]\right]}{v_k^{c\prime}}}.$$

The risk measure whose gradient yields the appropriate marginal cost of risk for assets can again be shown to be a conditional expectation calculated under a transformed measure:

$$\rho\,(A)=\mathbb{E}^{\widetilde{\mathbb{P}}}\left[A|I>A\right]$$

where the transformed measure is related to the physical probability measure $\mathbb{P}$ by the density:

$$\frac{\partial\widetilde{\mathbb{P}}}{\partial\mathbb{P}}=\frac{\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}}\,U_k^{u\prime}\,\frac{I_k^u}{I^u}\right]}{v_k^{u\prime}}+\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}}\,U_k^{c\prime}\,\frac{I_k^c}{I^c}\right]}{v_k^{c\prime}}}{\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{I^c<A<I\}}\,U_k^{u\prime}\,\frac{I_k^u}{I^u}\right]}{v_k^{u\prime}}+\sum_k\frac{\mathbb{E}\left[\mathbf{1}_{\{A<I^c\}}\,U_k^{c\prime}\,\frac{I_k^c}{I^c}\right]}{v_k^{c\prime}}}.$$

3 Comparison to Conventional Risk Measurement

Similarly to Bauer and Zanjani (2016), we consider a situation with N independent, identically distributed liability risks, $L_i\sim\text{Exp}(\nu)$, and homogeneous consumers that exhibit a constant absolute risk aversion α. In particular, we assume that their participation constraint is given by the situation without insurance, such that their reservation utility is:

$$\gamma=\mathbb{E}\left[-\exp\left\{-\alpha\,(w-L_N)\right\}\right]=-e^{-\alpha\,w}\frac{\nu}{\nu-\alpha},\ \alpha<\nu, \tag{18}$$

and we write $L=\sum_{i=1}^N L_i$.

However, we assume that the insurer, in addition to pooling a fraction q of the consumers' risk in return for a per-policy premium p, now has the possibility to invest in a risk-free asset (with a period return of zero) *and* in a risky asset, at amounts b and a, respectively. More precisely, we assume that at the end of the period, the risky asset is worth $a\,S$, where $S\sim\text{Exp}(\mu)$ is independent of liability risk. Thus, the company faces the profit maximization problem:

$$\left\{\begin{array}{l}\max_{a,b,q,p}\left\{N\,p-\mathbb{E}\left[\min\left\{q\,L,a\,S+b\right\}+a[S-1]\right]-\tau\left[a+b\right]\right\}\\ \text{s.t. }\mathbb{E}[-\exp\left\{-\alpha\left(w-p-L_N+\min\left\{q\,L_N,\frac{a\,S+b}{q\,L}\,q\,L_N\right\}\right)\right\}]=\gamma.\end{array}\right. \tag{19}$$

We obtain:

Proposition 3.1. *We can represent the optimization problem* (19) *as:*

$$\max_{a,b,q}\left\{\begin{array}{l}\underbrace{N\,p(a,b,q)}_{\textit{Premiums}}-\underbrace{\frac{qN}{\nu}\Gamma_{N+1,\nu}\left(\frac{b}{q}\right)}_{\textit{Act. Val. Liab.}}-\underbrace{b\left(\tau+\bar{\Gamma}_{N,\nu}\left(\frac{b}{q}\right)\right)-a(1+\tau)}_{\textit{Capital Cost}}\\ +\underbrace{\frac{a}{\mu}\left[\Gamma_{N,\nu}\left(\frac{b}{q}\right)+e^{\mu\,b/a}\frac{\nu^N}{(\nu+\mu q/a)^N}\bar{\Gamma}_{N,\nu+\mu q/a}\left(\frac{b}{q}\right)\right]}_{\textit{Capital Gains}}\end{array}\right\},$$

where $\Gamma_{n,x}$ is the cdf of the Gamma(N,x) distribution, $\bar{\Gamma}=1-\Gamma$, and the premium function is given by:

$$p(a,b,q)=-\frac{1}{\alpha}\log\left\{\begin{array}{l}\frac{\nu-\alpha}{\nu-\alpha(1-q)}\Gamma_{N-1,\nu}\left(\frac{b}{q}\right)+e^{\mu b/a}\frac{\nu-\alpha}{\mu\,q/a+\nu-\alpha(1-q)}\left(\frac{\nu}{\nu+\mu\,q/a}\right)^{N-1}\bar{\Gamma}_{N-1,\nu+\mu q/a}\left(\frac{b}{q}\right)\\ -e^{(\alpha(1-q)-\nu)b/q}\left(\frac{\nu}{\alpha(1-q)}\right)^{N-1}\left[\frac{\nu-\alpha}{\nu-\alpha(1-q)}-\frac{\nu-\alpha}{\mu q/a+\nu-\alpha(1-q)}\right]\Gamma_{N-1,\alpha(1-q)}\left(\frac{b}{q}\right)\\ +(\nu-\alpha)\,\mu\,(N-1)!\,e^{-\nu b}\sum_{k=0}^{\infty}\frac{\alpha^k}{(N-1+k)!}\sum_{j=0}^{N-1}\frac{(N+k-j-1)!}{(N-1-j)!\,j!}\nu^{j-k-}\\ \qquad\times\int_0^{\infty}(as+b)^j e^{-(\mu+a\nu)\,s}\,\overline{\Gamma}_{N+k-j,\nu}((as+b)(1-q)/q)\,ds\end{array}\right\}.$$

To assess the differences from the economic approach pursued here and conventional approaches to risk pricing and performance measurement based on risk measures and netting assets and liabilities, we compare the marginal cost of asset and liability risks in the two settings.

In the conventional setting, the starting point is an optimization problem similar to (19) where the company maximizes profits $\Pi(a,b,\{q_i\})$ less capital costs $\tau\times(a+b)$ (see Eq. (3) in Section 2), but rather than the including the counterparty participation constraints ((2) in the general model and (18) here), we assume assets have to satisfy the constraint from an exogenously supplied, homogeneous risk measure ρ:

$$\left\{\begin{array}{l}\Pi(a,b,\{q_i\})-\tau^*\times(a+b),\\ \text{s.t. }\rho\left(I+a(1-S)\right)\leq(a+b),\end{array}\right. \tag{20}$$

where q_i denotes the exposure to loss L_i, $I=\sum_i q_i L_i$, and $(1-S)$ is the relative loss on risky investment. Working with the optimality conditions and assuming existence of a solution, we immediately obtain for the marginal cost of asset and liability risk:

$$q_i\,\frac{\partial\Pi}{\partial q_i}=\tau^*\times q_i\,\frac{\partial\rho}{\partial q_i}\text{ and }a\,\frac{\partial\Pi}{\partial a}=\tau^*\times a\,\frac{\partial\rho}{\partial a}, \tag{21}$$

where of course $\sum_i q_i\,\frac{\partial\Pi}{\partial q_i}+a\,\frac{\partial\Pi}{\partial a}=a+b$. Hence, we can express the marginal cost of both asset and liability risk via the cost of capital τ^* times the respective capital allocation—which is the derivative of the risk measure at the optimum.

We can derive the following expressions for the case of Value-at-Risk (VaR) and Expected Shortfall (ES):

Proposition 3.2. *In the conventional setting, we have:*

- *For* $\rho = VaR_{\mathbb{P}(I>aS+b)}$:

$$\begin{aligned} q_i\,\frac{\partial\Pi}{\partial q_i} &= q\,\frac{1}{\nu+{}^{\mu q}/_a}\,\frac{\bar\Gamma_{N+1,\nu+{}^{\mu q}/_a}({}^b/_q)}{\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^b/_q)}\times\tau^*, \\ a\,\frac{\partial\Pi}{\partial a} &= \left[(a+b) - N\,\frac{1}{\nu+{}^{\mu q}/_a}\,q\,\frac{\bar\Gamma_{N+1,\nu+{}^{\mu q}/_a}({}^b/_q)}{\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^b/_q)}\right]\times\tau^*. \end{aligned}$$

- *For* $\rho = ES_{\mathbb{P}(I>aS+b^*)}$:

$$\begin{aligned} q_i\,\frac{\partial\Pi}{\partial q_i} &= \frac{N\,q}{\nu}\,\frac{\bar\Gamma_{N+1,\nu}({}^{b^*}/_q) - e^{\mu\,b^*/a}\frac{\nu^{N+1}}{(\nu+{}^{\mu\,q}/_a)^{N+1}}\,\bar\Gamma_{N+1,\nu+{}^{\mu q}/_a}({}^{b^*}/_q)}{\bar\Gamma_{N,\nu}({}^{b^*}/_q) - e^{\mu\,b^*/a}\frac{\nu^{N}}{(\nu+{}^{\mu\,q}/_a)^{N}}\,\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^{b^*}/_q)}\times\tau^*, \\ a\,\frac{\partial\Pi}{\partial a} &= \Bigg(\frac{b^*\,e^{\mu\,b^*/a}\frac{\nu^{N}}{(\nu+{}^{\mu\,q}/_a)^{N}}\,\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^{b^*}/_q) - \frac{N\,q}{\nu}e^{\mu\,b/a}\frac{\nu^{N+1}}{(\nu+{}^{\mu\,q}/_a)^{N+1}}\bar\Gamma_{N+1,\nu+{}^{\mu\,q}/_a}({}^{b^*}/_q)}{\bar\Gamma_{N,\nu}({}^{b^*}/_q) - e^{\mu\,b^*/a}\frac{\nu^{N}}{(\nu+{}^{\mu\,q}/_a)^{N}}\,\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^{b^*}/_q)} \\ &\qquad +\; a({}^1/_\mu - 1)\Bigg)\times\tau^*. \end{aligned}$$

As is evident from equations (4) and (5), in the economic setting similarly, the marginal cost of a liability exposure i is given by the allocation terms ϕ times the cost of capital:

$$\begin{aligned} \underbrace{q_i\,\frac{\partial p_i^*}{\partial q_i} - \mathbb{E}\left[1_{\{I\le aS+b\}}\,I_i\right]}_{q_i\,\frac{\partial\Pi}{\partial q_i}} &= \phi_i^q\times\left[(a+b)\times[\tau+\mathbb{P}(I>(aS+b))] - a\,\mathbb{E}\left[1_{\{I\le aS+b\}}\,[S-1]\right]\right] \\ &= \phi_i^q\times\left[b\times[\tau+\mathbb{P}(I>(aS+b))] + a\times[\tau+1-\mathbb{E}[S\,1_{\{I\le aS+b\}}]\right]. \end{aligned}$$

However, there are two notable differences: First, the allocation terms add up to one, $\sum_i \phi_i^q$, without including asset risk. So in contrast to the conventional setting where assets and liabilities are treated alike, here the liabilities are treated separately—and, indeed, as shown in the previous section, the risk measure supporting the allocations ϕ_i^q different from that for considering assets. To make up for not considering the assets in the allocation, as the second major difference, the hurdle rate term is adjusted downwards for the gain from exposure in risky assets.

We can again derive closed-form expressions for the economic marginal cost of risk:

Proposition 3.3. *In the economic setting, we have:*

$$\begin{aligned} q_i\,\frac{\partial\Pi}{\partial q_i} &= \phi_i^q\times\left[b\times[\tau+\mathbb{P}(I>(aS+b))] + a\times[\tau+1-\mathbb{E}[S\,1_{\{I\le aS+b\}}]\right] \\ &= \frac{q}{N}\times\Bigg[(a+b)\;\left[\tau+\bar\Gamma_{N,\nu}({}^b/_q)\right] - {}^a/_\mu\,e^{{}^{\mu\,b}/_a}\,\frac{\nu^N}{(\nu+{}^{\mu q}/_a)^N}\,\bar\Gamma_{N,\nu+{}^{\mu q}/_a}({}^b/_q) \\ &\qquad -[{}^1/_\mu - 1]\,\Gamma_{N,\nu}({}^b/_q) - q\,N\,e^{{}^{\mu\,b}/_a}\,\frac{\nu^N}{(\nu+{}^{\mu q}/_a)^{N+1}}\,\bar\Gamma_{N+1,\nu+{}^{\mu q}/_a}({}^b/_q)\Bigg]. \end{aligned}$$

Hence, while in the conventional setting we obtain non-trivial allocations to assets and liabilities but have the same given cost of capital, the economic approach gives a trivial allocation

($^1/_N$ for the N i.i.d. risks) but a nontrivial form for the cost of capital that arises from profitable investment opportunities.

For the asset side, Equation (5) implies the allocation of cost-adjusted returns of risk-free and risky asset:

$$\begin{aligned} a\,\left[\mathbb{E}\left[I_{\{I\le aS+b\}}\,[S-1]\right]-[\tau+\mathbb{P}(I>(aS+b))]\right] &= \phi_a^a\times\Big[a\,\mathbb{E}\left[I_{\{I\le aS+b\}}\,[S-1]\right] \\ &\qquad -(a+b)\times[\tau+\mathbb{P}(I>(aS+b))]\Big], \\ b\,[0-[\tau+\mathbb{P}(I>(aS+b))]] &= \phi_b^a\times\Big[a\,\mathbb{E}\left[I_{\{I\le aS+b\}}\,[S-1]\right] \\ &\qquad -(a+b)\times[\tau+\mathbb{P}(I>(aS+b))]\Big] \end{aligned}$$

To evaluate the differences between the conventional and the economic approach, we can now evaluate the differences in the marginal cost. The relevance is that companies will have to evaluate the cost side when appraising whether the expected profits (the left-hand sides of the marginal cost equations) are sufficient to cover their costs. The economic approach delivers the correct costs given the underlying assumptions, so the question arises how the company will fare when relying on the short-cut approach via an exogenous risk measure.

3.1 Results

We start by determining optimal choices by (numerically) solving the optimization problem in Proposition 3.1. More precisely, we set the number of liabilities N; we fix parameters for liability and asset risk ν and μ, respectively; we assume an absolute risk aversion level for our counterparties α; and we specify the cost-of-capital parameter τ. In Table 1 and Figure 1, we show resulting optimal policies. In particular, we show the optimal total capital level $(a+b)$, the relative investment in risky assets $^a/_{a+b}$, and the optimal coverage level q.

The results in Table 1 are not surprising. We find that increasing risk aversion ceteris paribus will spark the company to increase their capital holdings, to decrease their investment in risky assets, and will yield a higher insurance coverage q. The same is evident from Figure 1. Capital $(a+b)$ and coverage q are increasing in risk aversion α, whereas the investment in risky assets $^a/_{a+b}$ decreases in α.

Similarly to changing risk aversion, adjusting the expected return in the risky asset $^1/_\mu - 1$ will change the optimal choices. From the table and Figure 1c, it is clear that the risky asset holding increases in the expected return. Moreover, we also find that the overall capital holding and the insurance coverage increases as risky assets become more attractive. The key reason is that in the allocation equations (4) and (5), the cost-of-capital term $\left[\tau+\mathbb{P}(I>A)-\mathbb{E}[\mathbf{1}_{\{I\le A\}}\bar{S}]\right]$ is net of the expected return on risky assets $\mathbb{E}[\mathbf{1}_{\{I\le A\}}\bar{S}]$. Hence, increasing the expected return will translate into lower capital costs, which in turn will make capital holding less expensive and, as a consequence, insurance more attractive.

We are interested in comparing the outcomes to a conventional risk measurement and allocation approach, where assets and liabilities are netted and where an exogenous risk measure is used for allocating capital. We consider VaR and ES, with the resulting allocation provided in Proposition

Parametrizations and Optimal Choices

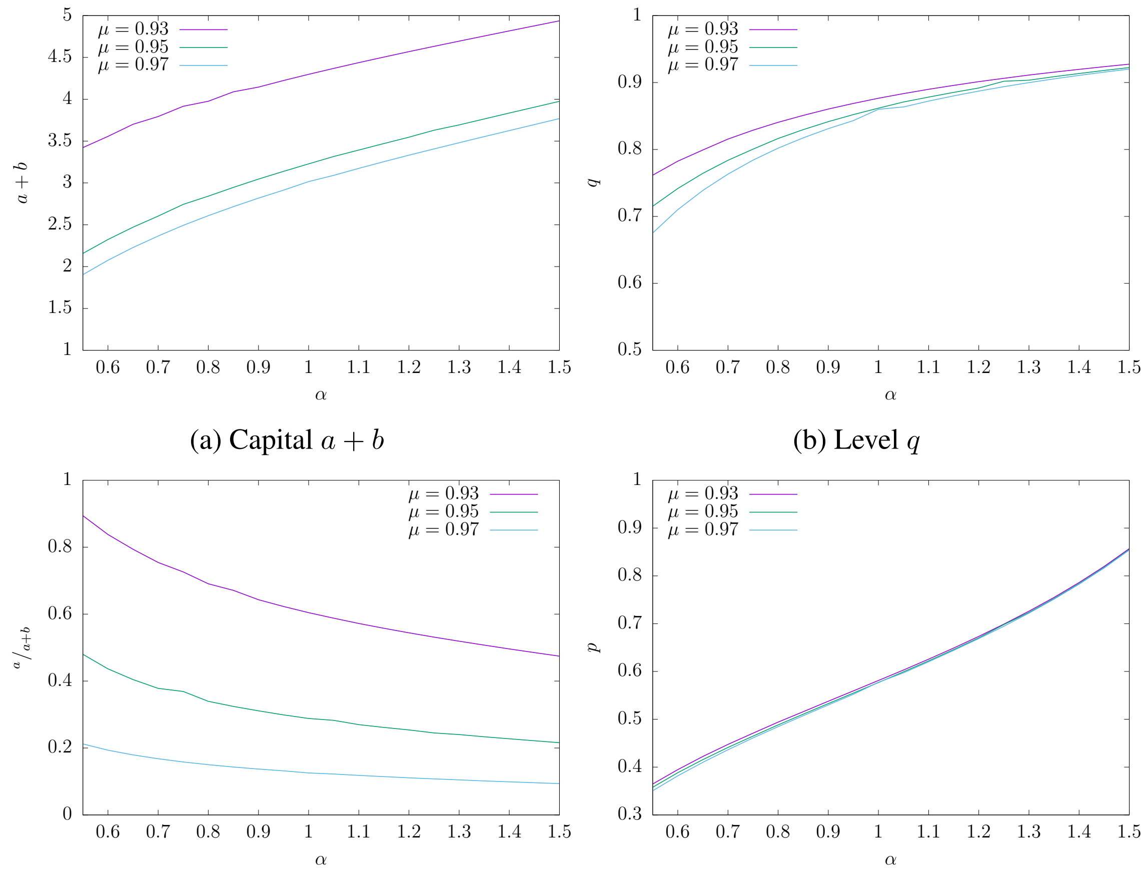
N	EL ($^1/_\nu$)	CoC (τ)	RiAv. (α)	Ret. ($^{1-\mu}/_\mu$)	Cap. ($a+b$)	RikP. ($^a/_{a+b}$)	Cov. (q)
5	0.5	0.1	0.7	7.53%	3.79	75.43%	81.55%
5	0.5	0.1	1.4	7.53%	4.82	49.61%	91.97%
5	0.5	0.1	0.7	5.26%	2.60	37.80%	78.39%
5	0.5	0.1	1.4	5.26%	3.84	22.75%	91.35%
5	0.5	0.1	0.7	3.09%	2.37	16.79%	76.32%
5	0.5	0.1	1.4	3.09%	3.63	9.89%	91.07%

Table 1: Parametrizations of the Exponential Losses model. The number of contracts (N), the expected loss (EL, $^1/_\nu$), the capital cost (CoC, τ), the risk aversion parameter (RiAv., α), and the return on risky assets (Ret., $^{1-\mu}/_\mu$) are set, and the optimal capital level (Cap., $a+b$), proportion in risky assets (RikP., $^a/_{a+b}$), and the coverage level (Cov., q) are determined as a solution of the optimization problem.

3.2. Note that here we adjust the threshold level such that total capital $(a+b)$ will be the same under all three approaches, and so will total capital costs $(a+b)\times\tau^*$. Table 2 provides risk allocations to liabilities, both in absolute terms ($q_i \frac{\partial \Pi}{\partial q_i}$) and relative to the total capital cost (in parentheses), for the three approaches: the economic approach according to Proposition 3.3, for VaR, and for ES. Of course, the remaining percentage is allocated to asset risk. Similarly, the left-hand side panels of Figure 2 plot the liability risk allocations for the economic approach and VaR.

We note that in all but one case within Table 2, namely for the high expected return of 7.53% and the high risk aversion level of 1.4, the conventional approaches allocate a large fraction of the total cost to liability risk relative to the economic approach. The average difference between the conventional and the economic approaches is roughly 2.7%, with the difference being as large as 13.1% for the highest expected return and the small risk aversion level for ES. Similarly, the left-hand side panels of Figure 2 demonstrate that the difference is larger for small risk aversion levels and becomes smaller for higher risk aversion levels—and even crosses in the case of $\mu = 0.93$.

A consequence to this observation is that the conventional approaches will frequently lead to a *higher* investment in risky assets. More precisely, relying on RAROC for internal steering, the company will expand risky investments as long as the risk-adjusted return exceeds the capital cost—and, in turn, retract the risky liability portfolio as long as the risk-adjusted return is below the capital cost. In the right-hand side panels of Figure 2, we determine the level of risky investment so that the marginal cost equates with the capital cost, holding all other parameters constant, and plot it relative to the optimal investment resulting from our economic approach shown in Figure 1c. In line with the argument above, the proportion in risky assets under VaR exceeds the economic outcome whenever the marginal cost for liability risk is larger—which as pointed out is frequently the case. The average difference is about 3.0%, but it can exceed 10% for small risk aversion levels.



(a) Capital $a + b$

(b) Level q

(c) Risky Assets $^a/_{a+b}$

(d) Premium p

Figure 1: Insurance level q, Total capital $(a + b)$, proportion of risky assets $^a/_{a+b}$, and premium p as a function of risk aversion α.

Hence, we conclude that relying on conventional approaches based on risk measures and net portfolio outcomes can lead to substantially different outcomes. Indeed, our results suggest that frequently the outcome is an over-investment in risky assets, which has interesting implications for risk measurement in practice.

4 Conclusion

In practice, the typical approach to pricing portfolio exposures in many financial institutions is based on RORAC or RAROC methodologies, where allocated capital is based on the gradient of a risk measure. Although this approach has been criticized (see, for example, Froot and Stein (1998); Erel, Myers, and Read (2015)) for lacking an economic foundation, the approach can be reconciled with economic fundamentals with appropriate calibration (Bauer and Zanjani, 2016). Once economic fundamentals are explicitly modeled, however, it becomes evident that the reality of such calibration is complex.

As shown in this paper, in a model where risk aversion is driven by counterparty preferences,

Parametrizations and Capital Cost Allocations

N	EL ($^1/_\nu$)	CoC (τ)	RiAv. (α)	Ret. ($^{1-\mu}/_\mu$)	LiabC. Ec.	LiabC. VaR	LiabC. ES
5	0.5	0.1	0.7	7.53%	0.64 (44.83%)	0.74 (51.38%)	0.83 (57.97%)
5	0.5	0.1	1.4	7.53%	0.68 (69.49%)	0.64 (64.85%)	0.64 (65.51%)
5	0.5	0.1	0.7	5.26%	0.81 (79.82%)	0.83 (82.56%)	0.86 (84.94%)
5	0.5	0.1	1.4	5.26%	0.70 (87.51%)	0.71 (89.04%)	0.71 (89.29%)
5	0.5	0.1	0.7	3.09%	0.83 (93.35%)	0.85 (95.47%)	0.86 (96.05%)
5	0.5	0.1	1.4	3.09%	0.72 (94.77%)	0.73 (97.30%)	0.74 (97.38%)

Table 2: Liability risk allocations of the Exponential Losses model. We present the allocation of capital costs to liability risk based on the economic approach, VaR, and ES for the different parameter sets considered in Table 1.

the economic impact of a risk exposure depends on its nature in two respects—1) whether it is an asset or liability, and 2) whether it collateralized or not. Our results show that risk evaluation based on "net exposure"—i.e., by focusing on the difference between assets and liabilities rather than their gross values—is fundamentally flawed and will not generally produce risk charges that are consistent with economic fundamentals. Similarly, ignoring the extent of the collateralization provisions in a liability exposure will lead to mismeasurement of the impact of the exposure on other counterparties.

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Appendix

A Proofs

Proof of Proposition 3.1. With (18), we can rewrite the constraint as:

$$\begin{aligned} \frac{\nu}{\nu-\alpha} &= e^{\alpha p}\,\mathbb{E}\left[\exp\left\{\alpha L_N - \alpha \min\left\{q\,L_N, \frac{a\,S+b}{L}\,L_N\right\}\right\}\right] \qquad (22)\\ \Rightarrow p(a,b,q) &= \frac{1}{\alpha}\left(\log\left\{\frac{\nu}{\nu-\alpha}\right\} - \log\left\{\mathbb{E}\left[\exp\left\{\alpha L_N - \alpha\min\left\{q\,L_N, \frac{a\,S+b}{L}\,L_N\right\}\right\}\right]\right\}\right)\end{aligned}$$

Write $L_{-N} = \sum_{i=1}^{N-1} L_i$ so that $L = L_{-N} + L_N$ and $L_{-N} \sim \text{Gamma}(N-1,\nu)$. Then we can write the expected value in the premium function (22) as:

$$\underbrace{\mathbb{E}\left[\exp\left\{\alpha\,(1-q)\,L_N\right\}\, I_{\{L_{-N}+L_N<(a\,S+b)/q\}}\right]}_{i.} + \underbrace{\mathbb{E}\left[\exp\left\{\left(\alpha\,L_N - \alpha(a\,S+b)\frac{L_N}{L_{-N}+L_N}\right)\right\}\, I_{\{L_{-N}+L_N\geq(a\,S+b)/q\}}\right]}_{ii.}.$$

For i., we have:

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\alpha\,(1-q)\,L_N\right\}\, I_{\{L_{-N}+L_N<(a\,S+b)/q\}}\right]\\ =&\int_0^\infty\int_0^\infty\int_0^\infty e^{\alpha(1-q)l} I_{\{l+i<a\,s/q+b/q\}}\,\nu\,e^{-\nu\,l}\,\mu\,e^{-\mu\,s}\,\frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\,dl\,di\,ds\\ =&\int_0^\infty \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\int_0^\infty e^{(\alpha(1-q)-\nu)\,l}\underbrace{\int_0^\infty I_{\{q(l+i)/a-b/a<s\}}\,\mu\,e^{-\mu\,s}\,ds}_{=e^{-\mu\,\max\{0,q(l+i)/a-b/a\}}}\,dl\,di\\ =&\int_0^\infty \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\int_0^\infty e^{(\alpha(1-q)-\nu)\,l}\,I_{\{l+i\leq b/q\}}\,dl\,di\\ &+\int_0^\infty \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\int_0^\infty e^{(\alpha(1-q)-\nu)\,l}\,I_{\{l+i>b/q\}}\,e^{-\mu\,(q(l+i)/a-b/a)}\,dl\,di\\ =&\int_0^{b/q} \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\int_0^{b/q-i} e^{(\alpha(1-q)-\nu)\,l}\,dl\,di\\ &+\int_0^\infty \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\int_{\max\{b/q-i,0\}}^\infty e^{(\alpha(1-q)-\nu)\,l-\mu\,(q(l+i)/a-b/a)}\,dl\,di\\ =&\int_0^{b/q} \frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-\nu\,i}\,\frac{1}{\nu-\alpha(1-q)}\left(1-e^{(\alpha(1-q)-\nu)(b/q-i)}\right)\,di\\ &+\int_0^\infty e^{\mu\,b/a}\,\frac{\nu^N}{(N-2)!}i^{N-2}\,e^{-(\nu+\mu\,q/a)\,i}\underbrace{\int_{\max\{b/q-i,0\}}^\infty e^{(\alpha(1-q)-\nu-\mu\,q/a)l}\,dl}_{=e^{(\alpha(1-q)-\nu-\mu\,q/a)\,\max\{b/q-i,0\}}/(\mu\,q/a+\nu-\alpha(1-q))}\,di\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\nu}{\nu-\alpha(1-q)} \underbrace{\int_0^{b/q} \frac{\nu^{N-1}}{(N-2)!} i^{N-2}\, e^{-\nu\, i}\, di}_{=\Gamma_{N-1,\nu}(b/q)} \\
& - \frac{e^{(\alpha(1-q)-\nu) b/q}\, \nu^N}{(\nu-\alpha(1-q))\,(\alpha(1-q))^{N-1}} \underbrace{\int_0^{b/q} \frac{(\alpha(1-q))^{N-1}}{(N-2)!} i^{N-2}\, e^{-\alpha(1-q)\, i}\, di}_{=\Gamma_{N-1,\alpha(1-q)}(b/q)} \\
& + \frac{e^{\mu\, b/a}\, \nu^N}{(\mu\, q/a+\nu-\alpha(1-q))\,(\nu+\mu\, q/a)^{N-1}} \underbrace{\int_{b/q}^{\infty} \frac{(\nu+\mu\, q/a)^{N-1}}{(N-2)!} i^{N-2}\, e^{-(\nu+\mu\, q/a)\, i}\, di}_{=1-\Gamma_{N-1,\nu+\mu\, q/a}(b/q)} \\
& + \frac{e^{(\alpha(1-q)-\nu)\, b/q} \nu^N}{(\mu\, q/a+\nu-\alpha(1-q))(\alpha(1-q))^{N-1}} \underbrace{\int_0^{b/q} \frac{(\alpha(1-q))^{N-1}}{(N-2)!} i^{N-2}\, e^{-\alpha(1-q) i}\, di}_{=\Gamma_{N-1,\alpha(1-q)}(b/q)} \\
= \quad & \frac{\nu}{\nu-\alpha(1-q)} \Gamma_{N-1,\nu}(b/q) + \frac{e^{\mu\, b/a}\, \nu^N}{(\mu\, q/a+\nu-\alpha(1-q))\,(\nu+\mu\, q/a)^{N-1}} \left(1-\Gamma_{N-1,\nu+\mu\, q/a}(b/q)\right) \\
& - \frac{e^{(\alpha(1-q)-\nu)\, b/q} \nu^N}{(\alpha(1-q))^{N-1}} \Gamma_{N-1,\alpha(1-q)}(b/q) \left(\frac{1}{\nu-\alpha(1-q)} - \frac{1}{\mu\, q/a+\nu-\alpha(1-q)}\right).
\end{aligned}
$$

For ii., we can write:

$$
\begin{aligned}
& \mathbb{E}\left[\exp\left\{\left(\alpha\, L_N - \alpha(a\, S+b)\frac{L_N}{L_{-N}+L_N}\right)\right\}\, \mathit{1}_{\{L_{-N}+L_N \geq (a\, S+b)/q\}}\right] \\
= \quad & \mathbb{E}\left[\exp\left\{(\alpha\,(L_{-N}+L_N) - \alpha\, b - \alpha\, a\, S)\, \frac{L_N}{L_{-N}+L_N}\right\}\, \mathit{1}_{\{L_{-N}+L_N \geq (a\, S+b)/q\}}\right],
\end{aligned}
$$

where $S \sim$ Exp(μ), $(L_{-N}+L_N) \sim$ Gamma(N, ν), and $\frac{L_N}{L_{-N}+L_N} \sim$ Beta$(1, N-1)$ are independent. Hence:

$$
\begin{aligned}
& \mathbb{E}\left[\exp\left\{\left(\alpha\, L_N - \alpha(a\, S+b)\frac{L_N}{L_{-N}+L_N}\right)\right\}\, \mathit{1}_{\{L_{-N}+L_N \geq (a\, S+b)/q\}}\right] \\
= \quad & \int_0^{\infty}\int_0^{\infty} \mathit{1}_{\{l \geq (as+b)/q\}}\, \mu\, e^{-\mu\, s}\, \frac{\nu^N}{(N-1)!} l^{N-1}\, e^{-\nu\, l} \\
& \qquad \times \underbrace{\int_0^1 e^{(\alpha\, l - \alpha b - \alpha\, a\, s)\, y}\, (N-1)\, (1-y)^{N-2}\, dy}_{=\text{mgf}_{\text{Beta}(1,N-1)}(\alpha\, l - \alpha b - \alpha a s)}\, dl\, ds \\
= \quad & \int_0^{\infty}\int_0^{\infty} \mathit{1}_{\{l \geq (as+b)/q\}}\, \mu\, e^{-\mu\, s}\, \frac{\nu^N}{(N-1)!} l^{N-1}\, e^{-\nu\, l} (N-1)! \sum_{k=0}^{\infty} \frac{(\alpha\, l - \alpha b - \alpha a s)^k}{(N-1+k)!}\, ds\, dl.
\end{aligned}
$$

Write $l^{N-1} = (l-(as+b)+(as+b))^{N-1} = \sum_{j=0}^{N-1}\binom{N-1}{j}(l-(as+b))^{N-1-j}(as+b)^j$ so that:

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{\left(\alpha L_N - \alpha(a\,S+b)\frac{L_N}{L_{-N}+L_N}\right)\right\} \mathit{1}_{\{L_{-N}+L_N \geq (a\,S+b)/q\}}\right] \\
= \;& \sum_{k=0}^{\infty}\sum_{j=0}^{N-1}\binom{N-1}{j}\int_0^\infty (as+b)^j\,\mu\,e^{-\mu\,s-\nu\,(as+b)}\frac{\alpha^k}{(N-1+k)!}\frac{(N+k-j-1)!}{\nu^{-j+k}} \\
&\times \underbrace{\int_{(as+b)/q}^\infty \frac{\nu^{N-j+k}}{(N+k-j-1)!}(l-(as+b))^{N-1+k-j}\,e^{-\nu\,(l-(as+b))}\,dl}_{=\int_{(as+b)(1-q)/q}^\infty \frac{\nu^{N-j+k}}{(N+k-j-1)!}\,l^{N-1+k-j}\,e^{-\nu\,l}\,dl = \overline{\Gamma}_{N+k-j,\nu}((as+b)(1-q)/q)}\;ds \\
= \;& \mu\,(N-1)!\,e^{-\nu\,b}\sum_{k=0}^\infty \frac{\alpha^k}{(N-1+k)!}\sum_{j=0}^{N-1}\frac{(N+k-j-1)!}{(N-1-j)!\,j!}\nu^{j-k} \\
&\times \int_0^\infty (as+b)^j e^{-(\mu+a\nu)\,s}\,\overline{\Gamma}_{N+k-j,\nu}((as+b)(1-q)/q)\,ds,
\end{aligned}$$

which delivers the premium function.

For the expected value in the objective function in (19), we obtain:

$$\begin{aligned}
&\mathbb{E}\left[\min\{q\,L, a\,S+b\}\right] \\
= \;& \mathbb{E}\left[q\,L\,\mathit{1}_{\{qL\leq a\,S+b\}}\right] + \mathbb{E}\left[(a\,S+b)\,\mathit{1}_{\{qL> a\,S+b\}}\right] \\
= \;& \int_0^\infty\int_0^\infty q\,l\,\mathit{1}_{\{q\,l\leq a\,s+b\}}\,\frac{\nu^N}{(N-1)!}\,e^{-\nu\,l}\,l^{N-1}\,\mu\,e^{-\mu\,s}\,ds\,dl \\
&+ \int_0^\infty\int_0^\infty (a\,s+b)\,\mathit{1}_{\{q\,l> a\,s+b\}}\,\frac{\nu^N}{(N-1)!}\,e^{-\nu\,l}\,l^{N-1}\,\mu\,e^{-\mu\,s}\,ds\,dl \\
= \;& \int_0^\infty q\,l\,\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\underbrace{\int_0^\infty \mathit{1}_{\{\max\{ql/a-b/a,0\}<s\}}\,\mu\,e^{-\mu\,s}\,ds}_{=e^{-\max\{ql/a-b/a,0\}\mu}}\;dl \\
&\int_0^\infty\left(b\underbrace{\int_0^\infty \mathit{1}_{\{ql>as+b\}}\mu\,e^{-\mu\,s}\,ds}_{=\mathit{1}_{\{l>b/q\}}\,(1-e^{-\mu(ql/a-b/a)})} + a\underbrace{\int_0^\infty s\,\mathit{1}_{\{ql>as+b\}}\mu\,e^{-\mu\,s}\,ds}_{=\mathit{1}_{\{l>b/q\}}\,\int_0^{ql/a-b/a}s\,\mu\,e^{-\mu\,s}\,ds}\right)\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl.
\end{aligned}$$

Integrating by parts, we obtain:

$$\begin{aligned}
\int_0^{ql/a-b/a} s\,\mu\,e^{-\mu\,s}\,ds &= \left[-s\,e^{-\mu\,s}\right]_0^{ql/a-b/a} + \int_0^{ql/a-b/a} e^{-\mu\,s}\,ds \\
&= -(ql/a-b/a)\,e^{-\mu\,(ql/a-b/a)} + 1/\mu - e^{-\mu\,(ql/a-b/a)}/\mu \\
&= 1/\mu - (1/\mu + ql/a - b/a)\,e^{-\mu\,(ql/a-b/a)}.
\end{aligned}$$

Thus:

$$
\begin{aligned}
&\mathbb{E}\left[\min\{q\,L, a\,S+b\}\right] \\
=\ & \int_0^\infty q\,l\,\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,\mathit{1}_{\{l<b/q\}}\,dl + \int_0^\infty q\,l\,\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,e^{-(ql/a-b/a)\,\mu}\,\mathit{1}_{\{l>b/q\}}\,dl \\
& +a\int_0^\infty \mathit{1}_{\{l>b/q\}}\left[1/\mu-(1/\mu-b/a)\,e^{-\mu\,(ql/a-b/a)}-q\,l\,e^{-\mu\,(ql/a-b/a)}/a\right]\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl \\
& +b\int_0^\infty \mathit{1}_{\{l>b/q\}}\left(1-e^{-\mu(ql/a-b/a)}\right)\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl \\
=\ & \frac{q\,N}{\nu}\underbrace{\int_0^{b/q}\frac{\nu^{N+1}}{N!}e^{-\nu\,l}\,l^N\,dl}_{=\Gamma_{N+1,\nu}(b/q)} + q\,e^{\mu\,b/a}\times \underbrace{\int_{b/q}^\infty \nu^N\frac{1}{(N-1)!}l^N\,e^{-(\nu+q\mu/a)\,l}\,dl}_{=\frac{N\,\nu^N}{(\nu+q\mu/a)^{N+1}}\underbrace{\int_{b/q}^\infty\frac{(\nu+q\mu/a)^{N+1}}{N!}l^N\,e^{-(\nu+q\mu/a)\,l}\,dl}_{=\bar{\Gamma}_{N+1,\nu+q\mu/a}(b/q)}} \\
& +a/\mu\underbrace{\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl}_{=\bar{\Gamma}_{N,\nu}(b/q)} - [a/\mu-b]\,e^{\mu b/a}\underbrace{\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-(\nu+\mu\,q/a)\,l}\,l^{N-1}\,dl}_{=\frac{\nu^N}{(\nu+\mu\,q/a)^N}\bar{\Gamma}_{N,\nu+\mu\,q/a}(b/q)} \\
& -q\,e^{\mu\,b/a}\underbrace{\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-(\nu+\mu\,q/a)\,l}\,l^N\,dl}_{=\frac{N\,\nu^N}{(\nu+\mu\,q/a)^{N+1}}\bar{\Gamma}_{N+1,\nu+\mu\,q/a}(b/q)} \\
& +b\underbrace{\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl}_{=\bar{\Gamma}_{N,\nu}(b/q)} - b\,e^{\mu\,b/a}\underbrace{\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-(\nu+\mu q/a)\,l}\,l^{N-1}\,dl}_{=\frac{\nu^N}{(\nu+\mu q/a)^N}\bar{\Gamma}_{N,\nu+\mu q/a}(b/q)}.
\end{aligned}
$$

Hence:

$$
\begin{aligned}
&\mathbb{E}\left[\min\{q\,L, a\,S+b\}\right] \\
=\ & \frac{q\,N}{\nu}\Gamma_{N+1,\nu}\left(\frac{b}{q}\right)+b\,\bar{\Gamma}_{N,\nu}\left(\frac{b}{q}\right)+\frac{a}{\mu}\left[\bar{\Gamma}_{N,\nu}\left(\frac{b}{q}\right)-e^{\mu\,b/a}\frac{\nu^N}{(\nu+\mu q/a)^N}\bar{\Gamma}_{N,\nu+\mu q/a}\left(\frac{b}{q}\right)\right].
\end{aligned}
$$

Plugging into (19) yields the result. □

Proof of Proposition 3.2. For Value-at-Risk, similarly as for the objective in Proposition 3.1 above:

$$
\begin{aligned}
\bar{p}=\mathbb{P}(qL>a\,S+b) &= \int_0^\infty\int_0^\infty \mathit{1}_{\{q\,l>as+b\}}\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,e^{-\mu\,s}\,\mu\,ds\,dl \\
&= \int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-\nu\,l}\,l^{N-1}\,dl - e^{\mu\,b/a}\int_{b/q}^\infty\frac{\nu^N}{(N-1)!}e^{-(\nu+\mu\,q/a)\,l}\,l^{N-1}\,dl \\
&= \bar{\Gamma}_{N,\nu}(b/q)-e^{\mu\,b/a}\frac{\nu^N}{(\nu+\mu q/a)^N}\bar{\Gamma}_{N,\nu+\mu q/a}(b/q).
\end{aligned}
$$

Hence, $\bar{p} = \mathbb{P}(qL - a(S-1) > a+b)$, or $\mathrm{VaR}_{\bar{p}}(qL - a(S-1)) = a+b$. For the allocation, in turn, we obtain:

$$\mathrm{VaR}_{\bar{p}}(qL - a(S-1)) = q \times \mathbb{E}[L|qL - a(S-1) = a+b] - a \times \mathbb{E}[S-1|qL - a(S-1) = a+b].$$

For the first term:

$$\begin{aligned}
\mathbb{E}[L\, 1_{\{qL-a(S-1)=a+b\}}] &= E[L\, 1_{\{S=(qL-b)/a\}}] \\
&= \int_{b/q}^{\infty} l\, \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1}\, \mu\, e^{-\mu/a\,(ql-b)}\, dl \\
&= N\, \mu\, e^{\mu\, b/a}\, \frac{\nu^N}{(\nu+\mu q/a)^{N+1}} \int_{b/q}^{\infty} \frac{(\nu+\mu q/a)^{N+1}}{N!} l^N\, e^{-(\nu+\mu\, q/a)\, l}\, dl \\
&= N\, \mu\, e^{\mu\, b/a}\, \frac{\nu^N}{(\nu+\mu q/a)^{N+1}}\, \bar{\Gamma}_{N+1,\nu+\mu q/a}(b/q),
\end{aligned}$$

and for the second term:

$$\begin{aligned}
\mathbb{E}[S\, 1_{\{qL-a(S-1)=a+b\}}] &= E[S\, 1_{\{S=(qL-b)/a\}}] \\
&= \int_{b/q}^{\infty} \left(\frac{ql}{a} - \frac{b}{a}\right) \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1}\, \mu\, e^{-\mu/a\,(ql-b)}\, dl \\
&= \frac{q}{a}\, N\, \mu\, e^{\mu\, b/a}\, \frac{\nu^N}{(\nu+\mu q/a)^{N+1}}\, \bar{\Gamma}_{N+1,\nu+\mu q/a}(b/q) \\
&\qquad - \frac{b}{a}\, \mu\, e^{\mu\, b/a}\, \frac{\nu^N}{(\nu+\mu q/a)^{N}}\, \bar{\Gamma}_{N,\nu+\mu q/a}(b/q).
\end{aligned}$$

We obtain for the density of $qL - a(S-1)$:

$$\begin{aligned}
f_{qL-a(S-1)}(x) &= -\frac{\partial}{\partial x} \mathbb{P}(qL - a(S-1) > x) \\
&= \frac{1}{a} \gamma_{N,\nu}((x-a)/q) + \mu\, e^{\mu\,(x-a)/a}\, \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}((x-a)/q) \\
&\qquad + \frac{1}{q} e^{\mu\,(x-a)/a}\, \frac{\nu^N}{(\nu+\mu q/a)^N} \gamma_{N,\nu+\mu q/a}((x-a)/q) \\
&= \mu\, e^{\mu\,(x-a)/a}\, \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}((x-a)/q).
\end{aligned}$$

Dividing the terms above by the density yields the result.

For Expected Shortfall, we have:

$$\mathbb{E}[(qL - a(S-1))\, 1_{\{qL-a(S-1)>a+b^*\}}] = q\, \mathbb{E}[L\, 1_{\{qL-a(S-1)>a+b^*\}}] - a\, \mathbb{E}[S\, 1_{\{qL-a(S-1)>a+b^*\}}] + a.$$

We obtain:

$$\begin{aligned}
\mathbb{E}[L\, 1_{\{qL-a(S-1)>a+b^*\}}] &= E[L\, 1_{\{(qL-b)/a>S\}}] \\
&= \int_{b/q}^{\infty} l\, \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1} \int_0^{(ql-b)/a} \mu\, e^{-\mu/s}\, ds\, dl \\
&= N \frac{1}{\nu} \bar{\Gamma}_{N+1,\nu}(b^*/q) - N\, e^{\mu\, b^*/a}\, \frac{\nu^N}{(\nu+\mu q/a)^{N+1}} \bar{\Gamma}_{N+1,\nu+\mu q/a}(b^*/q),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[S\, 1_{\{qL-a(S-1)>a+b^*\}}] &= E[S\, 1_{\{(qL-b)/a>S\}}] \\
&= \int_{b/q}^{\infty} \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1} \int_0^{(ql-b)/a} s\,\mu\, e^{-\mu/s}\, ds\, dl \\
&= \frac{b}{a} e^{\mu\, b^*/a} \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}(b^*/q) + \frac{1}{\mu}\bar{\Gamma}_{N,\nu}(b^*/q) \\
&\qquad - \frac{q}{a} e^{\mu\, b^*/a} N \frac{\nu^N}{(\nu+\mu q/a)^{N+1}} \bar{\Gamma}_{N+1,\nu+\mu q/a}(b^*/q) \\
&\qquad - \frac{1}{\mu} e^{\mu\, b^*/a} N \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}(b^*/q).
\end{aligned}$$

Dividing by $\mathbb{P}(qL - a(S-1) > a + b^*)$ yields the claim.

□

Proof of Proposition 3.3. From the previous proof, we have:

$$\mathbb{P}(qL > a\, S + b) = \bar{\Gamma}_{N,\nu}(b/q) - e^{\mu\, b/a} \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}(b/q),$$

and

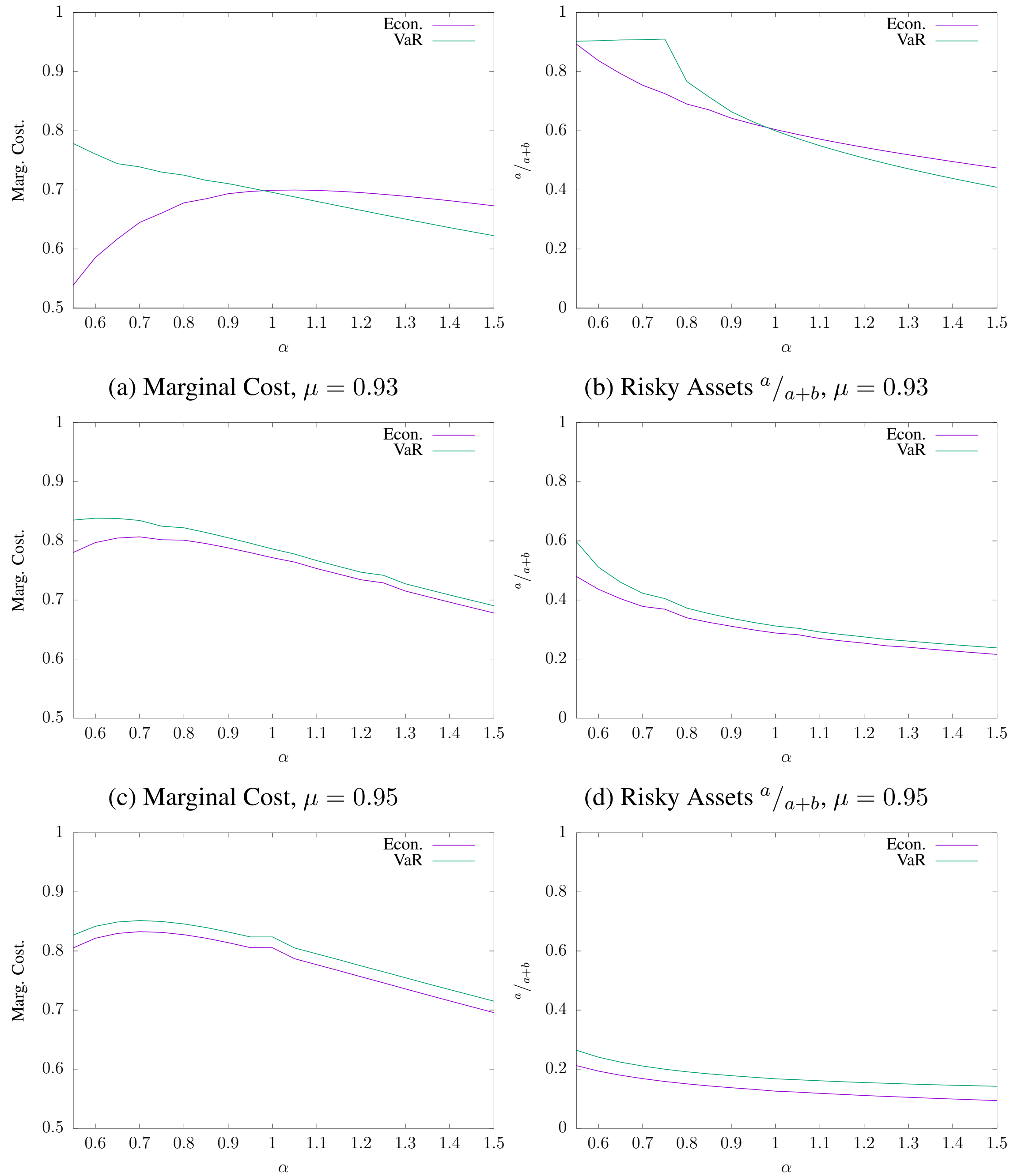
$$\begin{aligned}
\mathbb{E}[S\, 1_{\{qL\le aS+b\}}] &= \int_0^{\infty}\int_{ql-b/a \vee 0}^{\infty} \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1}\, s\,\mu\, e^{-\mu\, s}\, ds\, dl \\
&= \int_{b/q}^{\infty} \left(\frac{1}{\mu} + \frac{ql-b}{a}\right) e^{-\mu\frac{ql-b}{a}} \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1}\, dl \\
&\quad + \int_0^{b/q} \frac{1}{\mu} \frac{\nu^N}{(N-1)!} e^{-\nu\, l}\, l^{N-1}\, dl \\
&= \left(\frac{1}{\mu} - \frac{b}{a}\right) e^{\mu\, b/a} \frac{\nu^N}{(\nu+\mu q/a)^N} \bar{\Gamma}_{N,\nu+\mu q/a}(b/q) + \frac{1}{\mu}\Gamma_{N,\nu}(b/q) \\
&\quad + \frac{q}{a} e^{\mu\, b/a} \frac{\nu^N}{(\nu+\mu q/a)^{N+1}} N\, \bar{\Gamma}_{N+1,\nu+\mu q/a}(b/q).
\end{aligned}$$

Hence, we have for the marginal cost:

$$\begin{aligned}
q_i \frac{\partial \Pi}{\partial q_i} &= \frac{1}{N} \times \big((a+b)\,\tau - \mathbb{E}[a(S-1)\, 1_{\{qL\le aS+b\}}] + (a+b)\mathbb{P}(qL > a\, S + b)\big) \\
&= \frac{1}{N} \times \big(b\ (\tau + \mathbb{P}(qL > a\, S + b)) + a\left(\tau + 1 - \mathbb{E}[S\, 1_{\{qL\le aS+b\}}]\right)\big).
\end{aligned}$$

Plugging in the above yields the result.

□



(a) Marginal Cost, $\mu = 0.93$

(b) Risky Assets $^a/_{a+b}$, $\mu = 0.93$

(c) Marginal Cost, $\mu = 0.95$

(d) Risky Assets $^a/_{a+b}$, $\mu = 0.95$

(e) Marginal Cost, $\mu = 0.97$

(f) Risky Assets $^a/_{a+b}$, $\mu = 0.97$

Figure 2: Allocation to liability risk as a proportion of total capital as a function of risk aversion α for different parameters..